"Renormalization" Of Non Renormalizable Theories

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Abstract

A perturbative approach for non renormalizable theories is developed. It is shown that the introduction of an extra expansion parameter allows one to get rid of divergences and express physical quantities as series with finite coefficients. The method is demonstrated on the example of massive non abelian field coupled to a fermion field.

The presence of divergences is one of the basic problems of quantum field theory (QFT). The renormalization procedure handles these divergences only for a class of theories i.e renormalizable ones. It is not *a priory* clear that nonrenormalizable theories lack physical significance, and moreover, in spite of the fact that most of the fundamental interactions are described by renormalizable QFT-s, the problem of the quantum gravitation is still open — while Einstein's classical theory of gravitation has substantional success, the corresponding quantum theory is non renormalizable.

We share the opinion that the renormalizability is just a technical requirement and has nothing to do with the physical content of the QFT. A lot of people believe that in meaningful theories divergences arise due to the use of a perturbative expansion, and has been noted in various papers and various contexts [1],[2]. Of course not all of the non renormalizable theories are meaningful, which is also true for the renormalizable ones too. E.g. the scalar ϕ^3 theory is renormalizable for space-time dimensions up to six [3], but has an energy spectrum unbounded from below. On the other hand there exist non renormalizable theories which can be handled in some other approach (e.g. the four-fermion interaction in $2+1$ dimensions is non renormalizable if the conventional renormalization procedure is applied, but can be renormalized after performing a $1/N$ expansion with N being the number of flavours [4]).

Below we are going to present a method of extracting physical information out of the perturbative series of non renormalizable theories. For renormalizable ones it just coincides with the usual renormalization procedure and in that case alone can it be interpreted in terms of counter terms. We emphasize that described technique is unambigous.

Let us use the example of $SU(N)$ massive gauge field coupled to fermions to illustrate how the method works. This theory, being massive, suffers only from ultraviolet divergences, and is given by the Lagrangian:

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar{\psi}\hat{D}\psi - m_0\bar{\psi}\psi + M_0^2 A^{a\mu}A^a_\mu \tag{1}$$

where $G^a_{\mu\nu}$ is the gauge field strength tensor $(a = 1, ...N)$ and $\hat{D}$ is the covariant derivative with coupling g_0. This theory is not renormalizable [5], and we will work in terms of bare parameters using dimensional regularization $(n \equiv 4 + 2\epsilon)$. The Feynman rules for this model

may be derived in the standard manner. Simple power counting shows that the result of any diagram can be written in the following form:

$$\sum C_{ij}(\epsilon)\left(\frac{g_0^2}{\epsilon}\right)^i g_0^j \tag{2}$$

where the coefficients $C_{ij}(\epsilon)$ are expandable as positive power series of ϵ.

Let us proceed along the lines of the usual renormalization procedure. To make our method more transparent we avoid any numerical results of calculations.

We define the physical masses of vector meson and fermion as the pole positions of their propagators and express m_0 and M_0 in terms of physical masses m and M (mass renormalization). The wave function renormalization constants are defined as residues at the poles.

The amputated Green's function $< 0|T(\bar{\psi}A^a_\mu\psi)|0 >$ after mass and wave function renormalization takes the form (Fermion legs are on mass shell and q is a momentum of the vector field):

$$i\Gamma^a_{ij,\mu} = it^a_{ij}\left[A(q^2)\gamma_\mu + B(q^2)\sigma_{\mu\nu}q^\nu\right] \tag{3}$$

where

$$A(q^2) = a_0 g_0 + g_0^3\left[\frac{a_1(q^2)}{\epsilon} + a_2(q^2)\right] + ... \tag{4}$$

$$B(q^2) = b_0(q^2)g_0^3 + g_0^5\left[\frac{b_1(q^2)}{\epsilon} + b_2(q^2)\right] + g_0^7\left[\frac{b_3(q^2)}{\epsilon^2} + \frac{b_4(q^2)}{\epsilon} + b_5(q^2)\right] + ... \tag{5}$$

Note that the ϵ-dependence not shown explicitly is regular. (Evidently, A and B depend on all parameters of the theory). t^a_{ij} denote the group generators.

We introduce renormalized coupling constant as:

$$g = A(Q^2)/a_0 \tag{6}$$

Here Q^2 is the normalization point. Expressing g_0 from (6) we will have:

$$g_0 = g - g^3\left[\frac{a_1(Q^2)}{a_0\epsilon} + a_2(Q^2)/a_0\right] + ... \tag{7}$$

Substituting (7) into (5) we get:

$$B(q^2) = b_0(q^2)g^3 + g^5\frac{B_1(q^2,Q^2)}{\epsilon} + g^5 B_2(q^2,Q^2) + g^7\left[\frac{B_3(q^2,Q^2)}{\epsilon^2} + \frac{B_4(q^2,Q^2)}{\epsilon} + B_5(q^2,Q^2)\right] + ... \tag{8}$$

Let us introduce a "related" function

$$B^*(q^2) = b_0(q^2)g^3 + g^3x^2\frac{B_1(q^2,Q^2)}{\epsilon} + g^5 B_2(q^2,Q^2) + g^3x^4\frac{B_3(q^2,Q^2)}{\epsilon^2} +$$

$$+ g^5x^2\frac{B_4(q^2,Q^2)}{\epsilon} + g^7 B_5(q^2,Q^2) + ... \tag{9}$$

In (8) every inverse power of ϵ is accompanied by a g^2. One can rewrite (8) as series of g and $\frac{g^2}{\epsilon}$. (9) has been obtained by replacing this power of g^2 with x^2. Evidently, substituting $x = g$ into (9) we recover (8).

Extracting x^2 iteratively from (9) (x^2 is extracted at the point $q^2 = Q^2$, although we could take any other normalization point) we get:

$$x^2 = \epsilon \left[\alpha - \frac{B_4(Q^2, Q^2)}{B_1(Q^2, Q^2)} \alpha g^2 - \frac{B_3(Q^2, Q^2)}{B_1(Q^2, Q^2)} \alpha^2 + ... \right] \tag{10}$$

Here α is defined by the expression:

$$\alpha(Q^2) \equiv \frac{B^*(Q^2) - b_0(Q^2) g^3 - g^5 B_2(Q^2, Q^2) - g^7 B_5(Q^2, Q^2) - ...}{B_1(Q^2, Q^2) g^3} \tag{11}$$

From (2) it follows that after renormalization of the wave function and masses and substituting (7) for g_0 any physical cross section takes the form:

$$\sigma_i = \sum C^i_{ml} \left(\frac{g^2}{\epsilon} \right)^m g^l \tag{12}$$

Now consider a particular physical process $ff \to ff$. The cross section has the form:

$$\sigma = s_0 g^4 + g^6 \left[\frac{s_1}{\epsilon} + s_2 \right] + ... \tag{13}$$

Following previously define a "related" function

$$\sigma^* = s_o g^4 + g^4 x^2 \frac{s_1}{\epsilon} + g^6 s_2 + ... \tag{14}$$

Substitution of equation (10) into (14) gives:

$$\sigma^* = s_0 g^4 + s_1 g^4 \alpha + s_2 g^6 + ... \tag{15}$$

The $\epsilon \to 0$ limit is non divergent for (15), and in fact it is an expression for σ. So, for σ we get a finite series in terms of g and α. Analogously we produce finite expressions for all physical quantities.

To better understand the approach let us consider one simple example:

Suppose we have two functions f_1 and f_2 each given by series with divergent coefficients (in the $\epsilon \to 0$ limit):

$$f_1 = -\frac{g^3}{\epsilon} + \frac{g^5}{\epsilon} + \frac{1}{2} \frac{g^5}{\epsilon^2} + ... \tag{16}$$

$$f_2 = 1 + g + \frac{g^2}{\epsilon} - \frac{g^4}{\epsilon} + ... \tag{17}$$

Note that k-th inverse power of ϵ goes together with at least the k-th power of g^2. We again define "related" functions (in each term containing ϵ^{-k}, g^{2k} is replaced by x^{2k}), so:

$$f_1^* \equiv -g \frac{x^2}{\epsilon} + g^3 \frac{x^2}{\epsilon} + \frac{g}{2} \frac{x^4}{\epsilon^2} + ...$$

$$f_2^* \equiv 1 + g + \frac{x^2}{\epsilon} - g^2\frac{x^2}{\epsilon} + ... \tag{18}$$

Express x^2 iteratively from (18) as a power series of g and $\alpha^* \equiv f_2^* - 1 - g$ and substitute it into the expression of f_1^*. It is easy to see that the divergences disappear. We get:

$$x^2 = \epsilon(\alpha^* + \alpha^* g^2 + ...)$$

$$f_1^* = -(g\alpha^* - \frac{g}{2}\alpha^{*2} + ...) \tag{19}$$

The right hand side of (19) is the expansion of

$$f_1^* = -glog(1 + \alpha^*) = -glog(f_2^* - g) \tag{20}$$

Note that the same relation exist between f_1 and f_2. Indeed we have obtained (16) and (17) by "regularizing" and expanding the following functions:

$$f_1(g) = glogg^2 \to glog\frac{\frac{g^4}{\epsilon} + 1}{\frac{g^2}{\epsilon} + 1} \tag{21}$$

$$f_2(g) = g + \frac{1}{g^2} \to g + \frac{\frac{g^2}{\epsilon} + 1}{\frac{g^4}{\epsilon} + 1} \tag{22}$$

So we have recovered the existing relation (20) between f_1 and f_2 starting from series with divergent coefficients.

We would like to note that although initially in (21) and (22) we had a dependence on one parameter g, the expansion with finite coefficients became possible only after the introduction of an extra (not independent) parameter α.

So, for our example of massive vector field coupled to the fermion field, we have expressed physical quantities in terms of *two finite* parameters as series with finite coefficients. Before concluding that these series have any status one has to show that they are at least asymptotical. The situation is quite analogous to conventional renormalization procedure, where renormalizability of the theory does not mean that the theory is consistent. One should investigate whether the final series are (at least) asymptotical.

Although we have not investigated the problem of consistency for our example of a vector field coupled to fermionic one, it serves at least as an illustration of the suggested method as well as ϕ^4 theory for conventional renormalization.

To summarize, let us recall the steps we have made: First of all the usual mass and wave function renormalizations were performed. Next, the bare coupling g_0 was expressed in terms of an effective coupling constant g. Now, if we substitute g_0 expressed by g into physical quantities and the divergences disappear we know that the theory is renormalizable. In the non renormalizable case the divergences still survive and enter only via the combination g^β/ϵ (where β is fixed for given theory and can be calculated by simple power counting). So we can consider g^β as an independent parameter (in fact for mathematical rigour we have introduced "related" functions by replacing g^β with x^β) and express it as power series in g and another

(finite) effective ‘coupling constant’. All physical quantities are expressed in these two constants as series with finite coefficients. Of course the validity of this series will depend on the theory under consideration. The suggested method coincides with ordinary renormalization for renormalizable theories and involve the introduction of an extra effective parameter for non renormalizable ones.

The proposed method is easily applied within the framework of dimensional regularization, and it is not difficult to check that the method may be applied with other regularization schemes.

Finally we want to point out that we have applied our method to Einstein's theory of gravitation coupled to matter fields, and detail this most interesting case in a separate paper.

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References

[1] G.Parisi, Nucl.Phys. **B100**, (1975).

[2] N.Nakanishi and I.Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity,* World Scientific Lecture notes in Physics, v.27, World Scientific, 1990.
K.Symanzik, Commun.Math.Phys. **95**, p.445, (1984).
G.Parisi, Nucl.Phys. **B254**, p.58, (1985).
R.Jackiw and S.Templeton, Phys.Rev. **D23**, 2291, (1981).

[3] J.C.Collins, *Renormalization,* Cambridge University Press, 1984.

[4] B.Rosenstein, B.J.Warr and S.Park, Phys.Rev.Lett., **v62**, p.1433 (1989).

[5] C.Itzykson and J.-B.Zuber, *Quantum Field Theory,* McGraw-Hill book Co, 1985.